Coriolis and Centrifugal Forces

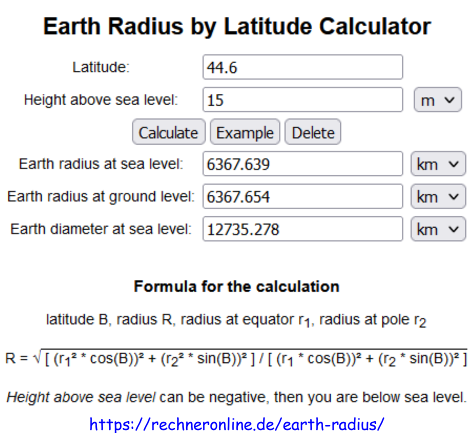
The equation describing the effective force on a particle on the surface of the Earth while acted upon by external forces $\vec{S}$, is

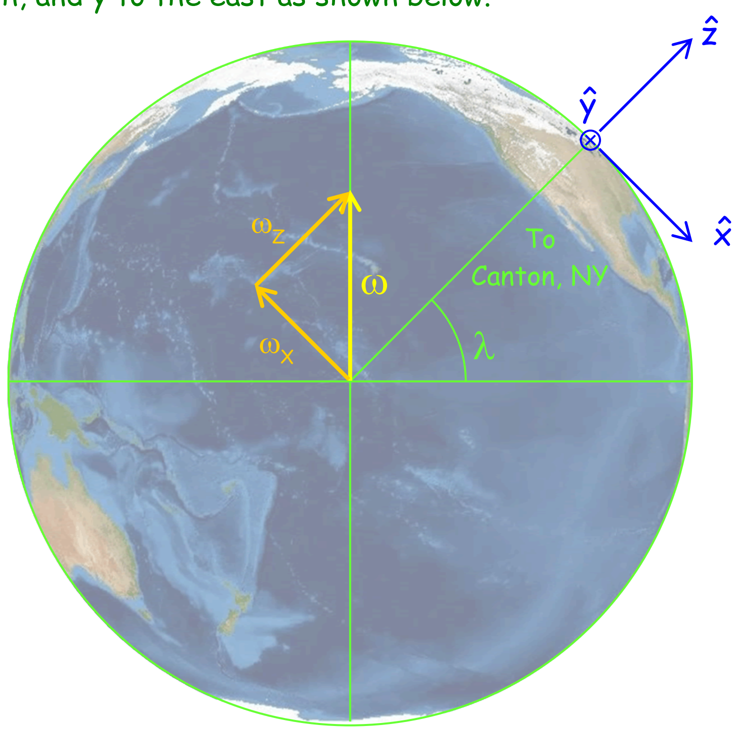
$$\vec{F}_{effective} = \vec{S} + m\vec{g}_0 - m\vec{\omega}_{\oplus} \times \left[\vec{\omega}_{\oplus} \times \left(\vec{R} + \vec{r}\right)\right] - 2m\left(\vec{\omega}_{\oplus} \times \vec{v}_r\right)$$

where R = the vector from the center of the Earth to the location, $\omega_{\oplus}$ is the angular velocity of the Earth, and v_r is the velocity of the particle as seen from the surface of the Earth.

Earth coordinate system

The standard coordinate system used on the surface of Earth has z upwards along the radial vector from Earth's Center, x to the south, and y to the east as shown below.





Since Earth is not spherical, the radius changes with latitude according to[1]

$$R_{\lambda} = \sqrt{\frac{\left(R_{Equator}^2 \cos\lambda\right)^2 + \left(R_{Pole}^2 \sin\lambda\right)^2}{\left(R_{Equator} \cos\lambda\right)^2 + \left(R_{Pole} \sin\lambda\right)^2}}$$

At Canton's latitude of 44.6°N, using $R_{Equator}$ = 6378.137 km and R_{Pole} = 6356.752 km gives R_{Canton} = 6367.639. Since Canton's elevation is 115 m, the radius from Earth's center to the ground in Canton is

$$R_{Canton} = 6367.754 \text{ km}.$$

[1] https://rechneronline.de/earth-radius/

The Coriolis Force

This force acts to deflect objects that are moving at a velocity of v_r with respect to the surface of the Earth:

$$\vec{F}_{Coriolis} = -2m\left(\vec{\omega}_{\oplus} \times \vec{v}_r\right)$$

In the Northern hemisphere, this deflects objects to their own right, in the southern hemisphere, it deflects them to their left. This can only be calculated for a specific velocity. However, we can get a general equation for the Coriolis force acting on a falling object by assuming that $\vec{v}_r = v_{fall}\hat{z} = -gt\hat{z}$.

$$\left[\vec{\omega}_{\oplus} \times v_{fall}\hat{z}\right] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\omega_{\oplus}\cos\lambda & 0 & \omega_{\oplus}\sin\lambda \\ 0 & 0 & -gt \end{vmatrix} = -\left(\omega_{\oplus} gt\cos\lambda\right)\hat{y}$$

Since it's an acceleration in y, this is a differential equation that can be solved for the deflection distance, y.

$$\ddot{y} = \frac{d^2y}{dy^2} = 2\omega_{\oplus} gt\cos\lambda$$

$$y(t) = \tfrac{1}{3}\omega_{\oplus} gt^3\cos\lambda$$

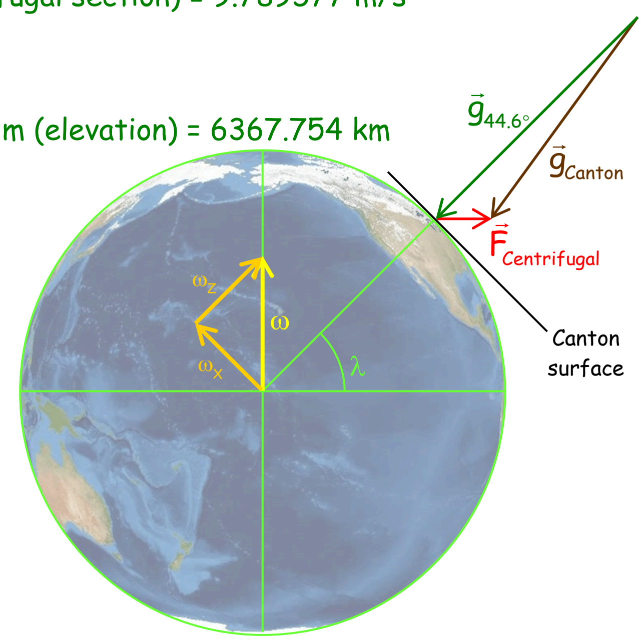
If it falls from a height, h, substituting $t = \sqrt{(2h/g)}$

$$y(h) = \frac{1}{3}\sqrt{\frac{8h^3}{g}}\,\omega_{\oplus}\cos\lambda$$

For Canton, using h = 100 m, gCanton (see the Centrifugal section) = 9.789577 m/s

The Centrifugal Force

For Canton, λ = 44.6°N, $R_{Canton} = R_{\odot,\,Canton}$ + 115 m (elevation) = 6367.754 km



$$\vec{F}_{Cf} = -m\vec{\omega}\times\left[\vec{\omega}\times\vec{R}_{Canton}\right]$$

$$\left[\vec{\omega}\times\vec{R}_{Canton}\right] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & 0 & R_{Canton} \end{vmatrix}$$

$$\left[\vec{\omega}\times\vec{R}_{Canton}\right] = -\left[\left(-\omega\cos\lambda\right)R_{Canton}\right]\hat{y}$$

$$\vec{\omega}\times\left[\vec{\omega}\times\vec{R}_{Canton}\right] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & R_{Canton}\omega\cos\lambda & 0 \end{vmatrix}$$

$$\vec{\omega}\times\left[\vec{\omega}\times\vec{R}_{Canton}\right] = \left(\omega\sin\lambda\right)\left(R_{Canton}\omega\cos\lambda\right)\hat{x} + \left(\omega\cos\lambda\right)\left(R_{Canton}\omega\cos\lambda\right)\hat{z}$$

$$\vec{\omega}\times\left[\vec{\omega}\times\vec{R}_{Canton}\right] = R_{Canton}\omega^2\cos\lambda\left[\underbrace{\left(\sin\lambda\right)\hat{x}}_{\text{deflection}} + \underbrace{\left(\cos\lambda\right)\hat{z}}_{\text{reduction}}\right]$$

The reduction of g in Canton is

$$\left|a_{Cf}\hat{z}\right| = \frac{\left|\vec{F}_{Cf,z}\right|}{m} = \left(R_{Canton}\omega^2\cos^2\lambda\right)$$

$$= \left(6367.8\times10^3\right)\left(7.27\times10^{-5}\right)^2\left(\cos^2\left(44.6\right)\right) = 0.0171\ \text{m/s}^2$$

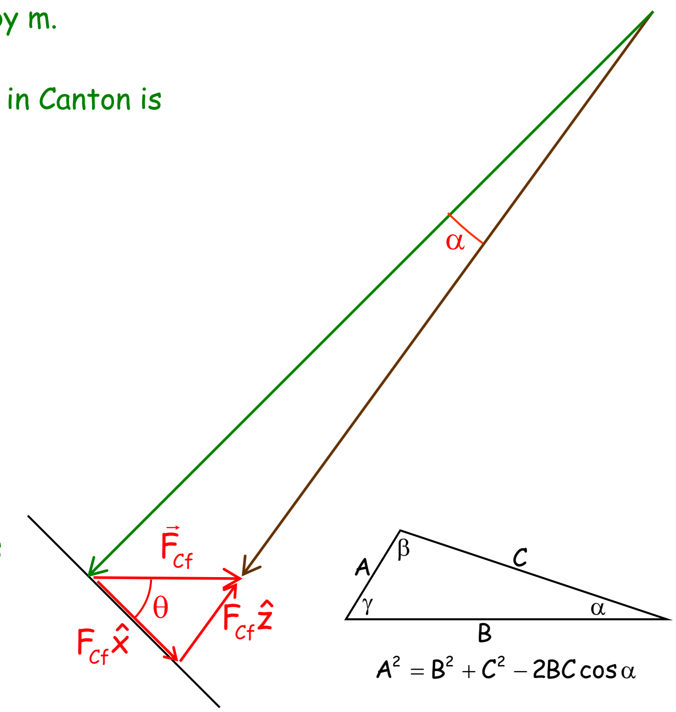
So that's a 0.174% reduction from g_0. A small effect!

The deflection of g in Canton is

$$\left|a_{Cf}\hat{x}\right| = \frac{\left|\vec{F}_{Cf,x}\right|}{m} = \left(R_{Canton}\omega^2\cos\lambda\sin\lambda\right)$$

$$= \left(6367.8\times10^3\right)\left(7.27\times10^{-5}\right)^2\left(\cos\left(44.6\right)\cos\left(44.6\right)\right) = 0.0168\ \text{m/s}^2$$

Find the angle away from the radial vector … angle of the pole-ward lean … of "vertical". Use only the acceleration of gravity … just divide by m.



The magnitude of the centrifugal acceleration in Canton is

$$\left|a_{Cf}\right| = \frac{\left|\vec{F}_{Cf}\right|}{m} = \sqrt{\left|a_{Cf,x}\right|^2 + \left|a_{Cf,z}\right|^2}$$

$$= \sqrt{\left(0.0168\ \text{m/s}^2\right)^2 + \left(0.0171\ \text{m/s}^2\right)^2}$$

$$= 0.02398\ \text{m/s}^2$$

Apply the law of cosines to find α:

$$a_{Cf} = g_0^2 + g_{Canton}^2 - 2g_0 g_{Canton}\cos\alpha$$

Giving the deflection angle of g_{Canton} from g_0 as

$$\cos\alpha = \frac{g_0^2 + g_{Canton}^2 - a_{Cf}}{2g_0 g_{Canton}}$$

Using the standard value of gravity and the Canton values from the previous page

$g_0 = 9.80665\ \text{m/s}^2$

$g_{Canton} = 9.80665 - 0.0171 = 9.78955\ \text{m/s}^2$

$a_{Cf} = 0.02398\ \text{m/s}^2$

$$\cos\alpha_{Canton} = \frac{\left(9.80665\right)^2 + \left(9.78955\right)^2 - \left(0.02398\right)^2}{2\left(9.80665\right)\left(9.78955\right)}$$

$$\alpha_{Canton} = \cos^{-1}\left(0.9999985\right) = 0.0983^\circ$$

So everything in Canton, plumb bobs, buildings, people, lean almost a tenth of a degree to the north in response to the centrifugal force. Cool!